A Note on Möbius Function and Möbius Inversion Formula of Fibonacci Cobweb Poset

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Abstract

The explicit formula for Möbius function of Fibonacci cobweb poset P is given here for the first time by the use of Kwaśniwski's definition of P in plane grid coordinate system [1].

1 Fibonacci cobweb poset

The Fibonacci cobweb poset P has been introduced by A.K.Kwaśniewski in [3, 4] for the purpose of finding combinatorial interpretation of fibonomial coefficients and their reccurence relation. At first the partially ordered set P (Fibonacci cobweb poset) was defined via its Hasse diagram as follows: It looks like famous rabbits grown tree but it has the specific cobweb in addition, i.e. it consists of levels labeled by Fibonacci numbers (the n-th level consist of F_n elements). Every element of n-th level ($n \geq 1, n \in \mathbf{N}$) is in partial order relation with every element of the $(n+1)$-th level but it's not with any element from the level in which he lies (n-th level) except from it.

2 The Incidence Algebra $\mathbf{I}(\mathbf{P})$

One can define the incidence algebra of $\mathbf{P}$ (locally finite partially ordered set) as follows (see [5, 6]):

$$\mathbf{I}(\mathbf{P}) = \{f : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbf{R}; \quad f(x,y) = 0 \quad unless \quad x \leq y\}.$$

The sum of two such functions f and g and multiplication by scalar are defined as usual. The product $H = f * g$ is defined as follows:

$$h(x,y) = (f * g)(x,y) = \sum_{z \in \mathbf{P}:\, x \leq z \leq y} f(x,z) \cdot g(z,y).$$

It is immediately verified that this is an associative algebra over the real field (associative ring).

The incidence algebra has an identity element $\delta(x,y)$, the Kronecker delta. Also the zeta function of $\mathbf{P}$ defined for any poset by:

$$\zeta(x,y) = \begin{cases} 1 & for \quad x \leq y \\ 0 & otherwise \end{cases}$$

is an element of $\mathbf{I}(\mathbf{P})$. The one for Fibonacci cobweb poset was expressed by δ in [1, 4] from where quote the result:

$$\zeta = \zeta_1 - \zeta_0 \tag{1}$$

where for $x, y \in \mathbf{N}$:

$$\zeta_1(x,y) = \sum_{k=0}^{\infty} \delta(x+k, y) \tag{2}$$

$$\zeta_0(x,y) = \sum_{k \geq 0} \sum_{s \geq 0} \delta(x, F_{s+1} + k) \sum_{r=1}^{F_s - k - 1} \delta(k + F_{s+1} + r, y). \tag{3}$$

The knowledge of ζ enables us to construct other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of $\mathbf{P}$. The one of them is Möbius function indispensable in numerous inversion type formulas of countless applications. It is known that the ζ function of a locally finite partially ordered set is invertible in incidence algebra and its inversion is the famous Möbius function μ i.e.:

$$\zeta * \mu = \mu * \zeta = \delta.$$

The Möbius function μ of Fibonacci cobweb poset $\mathbf{P}$ was presented for the first time by the present author in [2]. It was recovered by the use of the recurrence formula for Möbius function of locally finite partially ordered set $\mathbf{I}(\mathbf{P})$ (see [5]):

$$\begin{cases} \mu(x,x) = 1 \quad for \;\; all \;\; x \in \mathbf{P} \\[2ex] \mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) \end{cases} \tag{4}$$

The Möbius function of Fibonacci cobweb poset was there given by following formula:

$$\mu(x,y) = \begin{cases} 0 & x > y \\ 1 & x = y \\ 0 & F_{k+1} \leq x, y \leq F_{k+2} - 1;\ x \neq y;\ k \geq 3 \\ & \\ -1 & F_{k+1} \leq x \leq F_{k+2} - 1 < F_{k+2} \leq y \leq F_{k+3} - 1 \\ & \\ (-1)^{n-k} \prod_{l=k+1}^{n-1} (F_l - 1) & F_{k+1} \leq x \leq F_{k+2} - 1, \\ & F_{n+1} \leq y \leq F_{n+2} - 1;\ n > k+1, \end{cases} \tag{5}$$

where:

- the condition $F_{k+1} \leq x, y \leq F_{k+2} - 1;\ x \neq y;\ k \geq 3$ means that x, y are different elements of k-th level;
- the condition $F_{k+1} \leq x \leq F_{k+2} - 1 < F_{k+2} \leq y \leq F_{k+3} - 1$ means that x is an element of k-th level and y is an element of $(k+1)$-th level;
- the condition $F_{k+1} \leq x \leq F_{k+2} - 1,\ F_{n+1} \leq y \leq F_{n+2} - 1;\ n > k+1$ means that x is an element of k-th level and y is an element of n-th level.

The above formula allows us to find out the μ function matrix (see[2])but it is not good enough to be applied in compact form via Möbius inversion formula for cobweb poset. For this purpose more convenient, explicit formula is needed.

3 Plane grid coordinate system of P

In [1] A. K. Kwaśniewski defined cobweb poset P as infinite labeled graph oriented upwards as follows: Let us label vertices of P by pairs of coordinates: $\langle i, j \rangle \in \mathbf{N} \times \mathbf{N}$, where the second coordinate is the number of level in which the element of P lies (here it is the j-th level) and the first one is the number of this element in his level (from left to the right), here i . We shall refer, (following [1]) Φ_s as to the set of vertices (elements) of the s-th level, i.e.:

$$\Phi_s = \{\langle j, s \rangle,\ \ 1 \leq j \leq F_s\}, \quad s \in \mathbf{N}.$$

For example $\Phi_1 = \{\langle 1,1 \rangle\},\ \Phi_2 = \{\langle 1,2 \rangle\},\ \Phi_3 = \{\langle 1,3 \rangle, \langle 2,3 \rangle\}$,
$\Phi_4 = \{\langle 1,4 \rangle, \langle 2,4 \rangle, \langle 3,4 \rangle\},\ \Phi_5 = \{\langle 1,5 \rangle, \langle 2,5 \rangle, \langle 3,5 \rangle\ \langle 4,5 \rangle, \langle 5,5 \rangle\}$ ....

Then P is a labeled graph $P = (V, E)$ where

$$V = \bigcup_{p\geq 1} \Phi_p, \quad E = \{\langle\, \langle j,p\rangle, \langle q, p+1\rangle\, \rangle\}\,, \;\; 1 \leq j \leq F_p, \;\; 1 \leq q \leq F_{p+1}.$$

Now we can define the partial order relation on P as follows: let $x = \langle s,t\rangle, y = \langle u,v\rangle$ be elements of cobweb poset P. Then

$$(x \leq y) \Longleftrightarrow [(t < v) \vee (t = v \wedge s = u)].$$

4 The Möbius function and Möbius inversion formula on P

The above definition of P allows us to derive an explicit formula for Möbius function of cobweb poset P. To do this we can use the formula (5). Then for $x = \langle s,t\rangle,\;\; y = \langle u,v\rangle,\; 1 \leq s \leq F_t,\;\; 1 \leq u \leq F_v,\; t,v \in \mathbf{N}$ we have

$$\mu(x,y) = \mu\left(\langle s,t\rangle, \langle u,v\rangle\right) = \\ = \delta(t,v)\delta(s,u) - \delta(t+1,v) + \sum_{k=2}^{\infty} \delta(t+k,v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1) \tag{6}$$

where δ is the Kronecker delta defined by

$$\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

We can also derive more convenient then (1) formula for ζ function of P (the characterictic function of partial order relation in P):

$$\zeta(x,y) = \zeta\left(\langle s,t\rangle, \langle u,v\rangle\right) = \delta(s,u)\delta(t,v) + \sum_{k=1}^{\infty} \delta(t+k,v). \tag{7}$$

The formula (6) enables us to formulate following theorem (see [5]):

Theorem 4.1. (Möbius Inversion Formula of cobweb P)
Let $f(x) = f(\langle s,t\rangle)$ be a real valued function, defined for $x = \langle s,t\rangle$ ranging in cobweb poset P. Let an element $p = \langle p_1, p_2\rangle$ exist with the property that $f(x) = 0$ unless $x \geq p$.

Suppose that

$$g(x)=\sum_{\{y\in P:\ y\leq x\}} f(y).$$

Then

$$f(x)=\sum_{\{y\in P:\ y\leq x\}} g(y)\mu(y,x).$$

But using coordinates of x,y in P i.e. $x=\langle s,t\rangle,\ y=\langle u,v\rangle$ if

$$g(\langle s,t\rangle)=\sum_{v=1}^{t-1}\sum_{u=1}^{F_v}\left(f(\langle u,v\rangle)\right)+f(\langle s,t\rangle)$$

then we have

$$f(\langle s,t\rangle)=\sum_{v\geq 1}\sum_{u=1}^{F_v} g(\langle u,v\rangle)\mu(\langle s,t\rangle,\langle u,v\rangle)=$$

$$=\sum_{v\geq 1}\sum_{u=1}^{F_v} g(\langle u,v\rangle)\left[\delta(v,t)\delta(u,s)-\delta(v+1,t)+\sum_{k=2}^{\infty}\delta(v+k,t)(-1)^k\prod_{i=v+1}^{t-1}(F_i-1)\right]. \tag{8}$$

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